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ON THE ROOTS OF THE BESSEL AND CERTAIN RELATED FUNCTIONS.

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I.—ROOTS OF $J_n(x)$.

The formulas of Sir G. G. Stokes for the roots of $J_0(x)$, $J_1(x)$ in the form of descending series are well known.[*] It is first proposed to obtain a similar expression for the roots of $J_n(x)$, n being unrestricted.

1. *Use of descending series for $J_n(x)$, with an auxiliary angle θ.*

Take the descending series[†]

$$\sqrt{\tfrac{1}{2}\pi x}\,J_n(x) = \cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\varphi_n(x) + \sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi)\,\psi_n(x), \tag{1}$$

in which

$$\varphi_n(x) = 1 - \frac{(1^2 - 4n^2)(3^2 - 4n^2)}{2!\,(8x)^2} + \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)(7^2 - 4n^2)}{4!\,(8x)^4} - \ldots,$$

$$\psi_n(x) = \frac{1^2 - 4n^2}{8x} - \frac{(1^2 - 4n^2)(3^2 - 4n^2)(5^2 - 4n^2)}{3!\,(8x)^3} + \ldots;$$

and let

$$\varphi_n(x) = R\cos\theta, \quad \psi_n(x) = R\sin\theta;$$

then

$$\sqrt{\tfrac{1}{2}\pi x}\,J_n(x) = R\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta), \tag{2}$$

and the equation $J_n(x) = 0$ becomes

$$\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = 0;$$

whence

$$x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta = \tfrac{1}{2}(2s - 1)\pi,$$

s being any integer.

$$\therefore \quad x = \beta + \theta, \tag{3}$$

where

$$\beta = \tfrac{1}{4}\pi(2n + 4s - 1). \tag{4}$$

2. *Series for $\tan\theta$ in powers of $\frac{1}{8x}$.*

Let $\frac{1}{8}x^{-1} = y$, $4n^2 = m$; then, from (2) and (1),

$$\tan\theta = -(m - 1)\,y\,\frac{1 - by^2 + dy^4 - \ldots}{1 - ay^2 + cy^4 - \ldots}$$
$$= -(m - 1)\,y\{1 + (a - b)\,y^2 + [a(a - b) + d - c]\,y^4 + \ldots\},$$

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[*] Camb. Phil. Trans., Vol. IX, or Math. and Phys. Papers, Vol. II, pp. 353, 355.

[†] Annals of Mathematics, Vol. VIII, No. 2, p. 60.

in which

$$a = \tfrac{1}{2}(m-1)(m-9), \quad b = \tfrac{1}{6}(m-9)(m-25),$$

$$c = \tfrac{1}{24}(m-1)(m-9)(m-25)(m-49),$$

$$d = \tfrac{1}{120}(m-9)(m-25)(m-49)(m-81),$$

and

$$a - b = \tfrac{1}{3}(m^2 + 2m - 99),$$

$$a(a-b) + d - c = \tfrac{2}{15}(m-9)(m^3 + 15m^2 - 81m - 5695);$$

$$\therefore \tan\theta = -(m-1)y - \tfrac{1}{3}(m-1)(m^2+2m-99)y^3$$

$$- \tfrac{2}{15}(m-1)(m-9)(m^3+15m^2-81m-5695)y^5 - \ldots. \qquad (5)$$

3. *Series for θ, $(= x - \beta)$, in powers of* $\dfrac{1}{8x}$.

Since

$$\theta = \tan\theta - \tfrac{1}{3}\tan^3\theta + \tfrac{1}{5}\tan^5\theta - \ldots,$$

and $\tan\theta$ is of the form

$$Ay + By^3 + Cy^5 + \ldots,$$

$$\therefore \quad \theta = Ay + (B - \tfrac{1}{3}A^3)y^3 + (C - A^2B + \tfrac{1}{5}A^5)y^5 + \ldots,$$

which, on replacing A, B, C by the coefficients in (6) and reducing, becomes

$$\theta = -(m-1)y - \tfrac{4}{3}(m-1)(m-25)y^3$$

$$- \tfrac{32}{5}(m-1)(m^2-114m+1073)y^5 - \ldots; \qquad (6)$$

$$\therefore x = \beta - \frac{m-1}{8x} - \frac{4(m-1)(m-25)}{3(8x)^3}$$

$$- \frac{32(m-1)(m^2-114m+1073)}{5(8x)^5} - \ldots. \qquad (7)$$

4. *Series for x in powers of* $\dfrac{1}{\beta}$.

From an equation of the form

$$x = \beta + \frac{p}{x} + \frac{q}{x^3} + \frac{r}{x^5} + \ldots,$$

Lagrange's theorem gives

$$x = \beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r - 4pq + 2p^3}{\beta^5} + \ldots.$$

Substituting for p, q, r, the coefficients $-\frac{1}{8}(m-1), -\frac{4}{3.8^3}(m-1)(m-25), \ldots,$ the reduced result may be written

$$x_n^{(s)} = \beta - \frac{m-1}{8\beta} - \frac{4(m-1)(7m-31)}{3(8\beta)^3}$$

$$-\frac{32(m-1)(83m^2-982m+3779)}{15(8\beta)^5} - \ldots,^* \tag{8}$$

in which $x_n^{(s)}$ represents the sth root (in order of magnitude) of the equation $J_n(x) = 0$, while β stands for $\frac{1}{4}\pi(2n-1+4s)$, and m is $4n^2$.

II.—ROOTS OF THE FIRST DERIVATIVE OF $J_n(x)$.†

5. *Descending series for $J_n'(x)$.*

Divide equation (1) by $x^{\frac{1}{2}}$, differentiate, multiply by $2x^{\frac{1}{2}}$, and rearrange; then

$$\sqrt{2\pi x}\, J_n'(x) = M\cos(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi) - N\sin(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi), \tag{9}$$

in which

$$M = \frac{2A_1 - A_0}{x} - \frac{2A_3 - 5A_2}{x^3} + \frac{2A_5 - 9A_4}{x^5} - \ldots,$$

$$N = 2A_0 - \frac{2A_2 - 3A_1}{x^2} + \frac{2A_4 - 7A_3}{x^4} - \ldots,$$

$A_0, A_1, -A_2, -A_3, \ldots$ being the coefficients of successive powers of x^{-1} in equation (1). The simplified expressions are

$$\left.\begin{aligned} M &= -(2m+6)y + \tfrac{1}{3}y^3(1+25-341+315) \\ &\quad -\tfrac{1}{60}y^5(1+15-6342+182510-1267659+1091475)+\ldots, \\ N &= 2 - y^2(m^2+14m-15) \\ &\quad +\tfrac{1}{12}y^4(1+28-1946+16092-14175)-\ldots, \end{aligned}\right\} \tag{10}$$

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* As it may sometimes be desirable to continue the series one step further, it is perhaps worth while to record that the next term is

$$-\frac{64(m-1)(6949m^3-153855m^2+1585743m-6277237)}{105(8\beta)^7}.$$

When n is 0, or 1, equation (8) reduces to Stokes's formulas; and when n is $\frac{3}{2}$ it gives the series for the roots of $J_{\frac{3}{2}}(x)$ obtained independently by Lord Rayleigh (Theory of Sound, Vol. I, p. 279).

In Professor Stokes's second formula the numerator 0.245835 should be 0.245270, a mistake which evidently arose from overlooking a "star" in the table of logarithms.

† The equation $J_n'(x) = 0$ is of hardly less importance in applied mathematics than the equation $J_n(x) = 0$. See Rayleigh's Theory of Sound, Vol. II, p. 266, Lamb's Hydrodynamics, p. 197, J. J. Thomson's Recent Researches in Electricity and Magnetism, p. 346.

in which m stands for $4n^2$, and y for $\frac{1}{8}x^{-1}$, the coefficients of the descending powers of m being " detached " for convenience.

Next, let an auxiliary angle θ' be defined by the equations

$$M = s \sin \theta' , \quad N = s \cos \theta' ; \tag{11}$$

then

$$\sqrt{2\pi x}\, J_n'(x) = - s \sin (x - \tfrac{1}{4}\pi - \tfrac{1}{2} n\pi - \theta') , \tag{12}$$

and the equation $J_n'(x) = 0$ becomes, as before,

$$x = \beta' + \theta' , \tag{13}$$

where

$$\beta' = \tfrac{1}{4}\pi (2n + 4s + 1) .$$

6. *Series for tan θ'.*

By (11) and (10), the reduced expression is

$$\tan \theta' = - y (m + 3) - \tfrac{1}{3} y^3 (1 + 13 + 211 - 225)$$
$$- \tfrac{2}{15} y^5 (1 + 25 + 658 + 11370 - 97779 + 85725) - \ldots . \tag{14}$$

7. *Series for θ', $(= x - \beta')$, in powers of y.*

The method of Art. 3 gives

$$x = \beta' - \frac{m+3}{8x} - \frac{4(1 + 46 - 63)}{3(8x)^3}$$
$$- \frac{32(1 + 185 - 2053 + 1899)}{5(8x)^5} \ldots . \tag{15}$$

8. *Series for x in powers of $\dfrac{1}{8\beta'}$.*

Applying Lagrange's theorem as in Art. 4, the final result is

$$x_n^{(s)} = \beta' - \frac{m+3}{8\beta'} - \frac{4(7m^2 + 82m - 9)}{3(8\beta')^3}$$
$$- \frac{32(83m^3 + 2075m^2 - 3039m + 3537)}{15(8\beta')^5} \ldots ;^{*} \tag{16}$$

in which $x_n^{(s)}$ represents the sth root (in order of magnitude) of the equation $J_n'(x) = 0$, m is $4n^2$, and β' is $\frac{1}{4}\pi(2n + 4s + 1)$.

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[*] It will be found that $n = 0$ in (16) gives the same result as $n = 1$ in (8), in accordance with the identity $J_0'(x) = - J_1(x)$.

A further test is furnished by the fact that when $n = \frac{1}{2}$, the equation $J_n'(x) = 0$ is equivalent to $\tan x = 2x$, as may be seen from (9), and that the independent solution of the latter equation agrees with (16). Similarly when $n = \frac{3}{2}$.

III.—ROOTS OF THE DERIVATIVE OF $x^{-\frac{1}{2}}J_n(x)$.*

9. It will be found that the M and N of Art. 5 are now to be replaced by

$$M' = \frac{A_1 - A_0}{x} - \frac{A_3 - 3A_2}{x^3} + \frac{A_5 - 5A_4}{x^5} - \ldots,$$

$$N' = A_0 - \frac{A_2 - 2A_1}{x^2} + \frac{A_4 - 4A_3}{x^4} - \ldots;$$

and that the series for tan θ' becomes

$$\tan\theta'' = -y(m+7) - \tfrac{1}{3}y^3(1 + 25 + 427 - 453)$$
$$- \tfrac{2}{15}y^5(1 + 45 + 1378 + 23490 - 158819 + 133905) - \ldots;$$

whence, as in Arts. 3, 7, is derived the series for θ'', $(= x - \beta')$, giving

$$x = \beta' - \frac{m+7}{8x} - \frac{4(1 + 70 - 199)}{3(8x)^3} - \frac{32(1 + 245 - 3693 + 4471)}{5(8x)^5} - \ldots,$$

and finally, by Lagrange's theorem, as in Art. 4,

$$x_n^{(s)} = \beta' - \frac{m+7}{8\beta'} - \frac{4(7m^2 + 154m + 95)}{3(8\beta')^3}$$
$$- \frac{32(83m^3 + 3535m^2 + 3561m + 6133)}{15(8\beta')^5} - \ldots .\dagger \qquad (17)$$

IV.—ROOTS OF THE SECOND BESSEL FUNCTION, AND OF ITS DERIVATIVE.

10. Take the descending series‡

$$\sqrt{\frac{2x}{\pi}}\,K_n(x) = \cos(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi)\,\varphi_n(x) + \sin(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi)\,\psi_n(x), \qquad (18)$$

in which $\varphi_n(x)$, $\psi_n(x)$ are the same as in Art. 1;

$$\therefore \sqrt{\frac{2x}{\pi}}\,K_n(x) = R\cos(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi - \theta),$$
$$= -R\cos(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta); \qquad (19)$$

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* The roots of the equation $\frac{d}{dx}[x^{-\frac{1}{2}}J_n(x)] = 0$ determine the possible wave lengths of a gas vibrating within a spherical envelope (Theory of Sound, Vol. II, p. 231).

† As a test, $n = \frac{1}{2}$ in (17) gives the same result as $n = \frac{3}{2}$ in (8), both being the solution of $\tan x = x$; and $n = \frac{3}{2}$ in (17) gives correctly the roots of $(2 - x^2)\tan x = 2x$, referred to on pp. 232-3, Theory of Sound, Vol. II.

‡ Annals of Mathematics, Vol. VIII, No. 2, p. 61.

and the equation $K_n(x) = 0$ gives

$$\cos(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = 0\,;$$

$$\therefore\quad x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta = \tfrac{1}{2}(2s-1)\pi\,;$$

$$\therefore\quad x = \beta'' + \theta\,,$$

where
$$\beta'' = \tfrac{1}{4}\pi(2n + 4s - 3)\,,$$

and θ is the same as in Sect. I. Hence the series in (8), which gives the roots of $J_n(x)$, gives also the roots of $K_n(x)$, when β'', $(= \beta - \frac{1}{2}\pi)$, is put for β.

11. *Derivative of* $K_n(x)$.

In the same way as in Art. 5 equations (9), (12) were derived from (1), so from (18) it follows that

$$2\sqrt{\frac{2x}{\pi}}\,K_n'(x) = -S\sin(x - \tfrac{3}{4}\pi - \tfrac{1}{2}n\pi - \theta')$$

$$= S\sin(x + \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta')\,; \qquad (20)$$

$\therefore$ the series in (16) gives also the roots of $K'_n(x)$, if β' be diminished by $\frac{1}{2}\pi$.

V.—ROOTS OF THE COMPLETE BESSEL FUNCTION, AND OF ITS DERIVATIVE.

12. In a large class of physical problems,* k is to be determined so that the complete function

$$AJ_n(kr) + BK_n(kr)$$

may vanish on the concentric circumferences $r = a$, $r = b$. In this case

$$AJ_n(ka) + BK_n(ka) = 0\,,$$

$$AJ_n(kb) + BK_n(kb) = 0\,;$$

$$\therefore\quad \frac{K_n(ka)}{J_n(ka)} = \frac{K_n(kb)}{J_n(kb)}\,.$$

Let $a = \rho b$, $\rho > 1$, and $kb = x$;

$$\therefore\quad \frac{K_n(x)}{J_n(x)} = \frac{K_n(\rho x)}{J_n(\rho x)}\,; \qquad (21)$$

but

$$\frac{K_n(x)}{J_n(x)} = \frac{\pi}{2}\tan(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta)\,, \qquad (22)$$

by (2) and (19);

$$\therefore\quad \tan(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta) = \tan(\rho x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta_1)\,,$$

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* See Byerly's "Fourier Series, etc.," pp. 230, 231, 234.

$$\therefore \quad \rho x - \theta_1 = x - \theta + s\pi ,$$

s being any integer. Substitute for θ the series in (6), and for θ_1 use the same series with x replaced by ρx;

$$\therefore \quad x = \frac{s\pi}{\rho - 1} + \frac{m-1}{8\rho x} + \frac{4(m-1)(m-25)(\rho^3-1)}{3(8\rho x)^3(\rho-1)}$$

$$+ \frac{32(m-1)(m^2-114m+1073)(\rho^5-1)}{5(8\rho x)^5(\rho-1)} + \ldots; \qquad (23)$$

whence, as in Art. 4,

$$x_n^{(s)} = \beta + \frac{p}{\beta} + \frac{q-p^2}{\beta^3} + \frac{r-4pq+2p^3}{\beta^5} + \ldots, \qquad (24)$$

wherein

$$\beta = \frac{s\pi}{\rho-1}, \ p = \frac{m-1}{8\rho},$$

$$q = \frac{4(m-1)(m-25)(\rho^3-1)}{3(8\rho)^3(\rho-1)}, \ r = \frac{32(m-1)(m^2-114m+1073)(\rho^5-1)}{5(8\rho)^5(\rho-1)}.$$

13. In certain problems in fluid motion* the function

$$AJ_n'(kr) + BK_n'(kr)$$

is to be treated as in Art. 12; and the equation

$$\frac{K_n'(x)}{J_n'(x)} = \frac{K_n'(\rho x)}{J_n'(\rho x)}$$

is by (12) and (20) equivalent to

$$\tan(x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta') = \tan(\rho x - \tfrac{1}{4}\pi - \tfrac{1}{2}n\pi - \theta_1').$$

Substitute for θ' the series in Art. 7, and for θ_1' the same with ρx used for x, and the result is of the same form as (24), with

$$p = \frac{m+3}{8\rho}, \quad q = \frac{4(m^2+46m-63)(\rho^3-1)}{3(8\rho)^3(\rho-1)},$$

$$r = \frac{32(m^3+185m^2-2053m+1899)(\rho^5-1)}{5(8\rho)^5(\rho-1)}.$$

14. *Wave lengths of a gas bounded by concentric spherical surfaces.*

This problem is the same extension of Section III as the last is of Section II; hence the series for θ'' in Art. 9 is to take the place of that for θ' in

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* As in computing the wave lengths of a gas or liquid bounded by two coaxal cylindrical surfaces.

Art. 7, and the result is again of the same form as (24), but

$$p = \frac{m + 7}{8\rho}, \quad q = \frac{4\,(m^2 + 70m - 199)\,(\rho^3 - 1)}{3\,(8\rho)^3\,(\rho - 1)},$$

$$r = \frac{32\,(m^3 + 245m^2 - 3693m + 4471)\,(\rho^5 - 1)}{5\,(8\rho)^5\,(\rho - 1)}.$$

GENERAL NOTE.

Some of the foregoing series for $x_n^{(s)}$ may for certain values of n and the lower values of s not be sufficiently convergent to give a very close approximation to the corresponding root. In such cases better approximations to the earlier roots may be found by interpolation from appropriate tables.